ESTIMATING SIZES OF A CONVEX BODY BY SUCCESSIVE DIAMETERS AND WIDTHS

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Abstract. The second theorem of Minkowski establishes a relation between the successive minima and the volume of a 0-symmetric convex body. Here we show corresponding inequalities for arbitrary convex bodies, where the successive minima are replaced by certain successive diameters and successive widths.

We further give some applications of these results to successive radii, intrinsic volumes and the lattice point enumerator of a convex body.

1. Introduction

Throughout this paper E^d denotes the d-dimensional euclidean space and the set of all convex bodies — compact convex sets — in E^d is denoted by $\mathcal{K}^d$. Further $\mathcal{K}_0^d$ denotes the 0-symmetric convex bodies, i.e. $K \in \mathcal{K}^d$ with $K = -K$. As usual $V(K)$ denotes the volume of a convex body and the set of all i-dimensional linear subspaces of E^d is denoted by $\mathcal{L}_i^d$. For $L \in \mathcal{L}_i^d$, $L^\perp$ denotes the orthogonal complement and for $K \in \mathcal{K}^d$, $L \in \mathcal{L}_i^d$ the orthogonal projection of K onto L is denoted by $K|L$.

The diameter and width of a convex body $K \in \mathcal{K}^d$ are denoted by $D(K)$ and $\Delta(K)$. For a detailed description of these functionals we refer to the book [BoF]. For any function Φ depending on the dimension we write $\Phi(M, A)$ for an affine plane A and $M \subset A$ to denote that Φ has to be computed with respect to the euclidean space A. With this notation we can define the following series of successive diameters and successive widths

Definition 1.1. For $K \in \mathcal{K}^d$ and $1 \le i \le d$ let

$$i)\ D_i^\pi(K) := \min_{L \in \mathcal{L}_i^d} D(K|L), \qquad ii)\ D_i^\sigma(K) := \min_{L \in \mathcal{L}_i^d} \max_{x \in L^\perp} D(K \cap (x + L)),$$

$$iii)\ \Delta_\pi^i(K) := \max_{L \in \mathcal{L}_i^d} \Delta(K|L; L), \qquad iv)\ \Delta_\sigma^i(K) := \max_{L \in \mathcal{L}_i^d} \max_{x \in L^\perp} \Delta(K \cap (x + L); x + L).$$

We obviously have $D_d^\pi(K) = D_d^\sigma(K) = \Delta_\pi^1(K) = \Delta_\sigma^1(K) = D(K)$, $D_1^\pi(K) = D_1^\sigma(K) = \Delta_\pi^d(K) = \Delta_\sigma^d(K) = \Delta(K)$ and these successive diameters and widths are

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continuous, monotone, homogeneous of degree 1 and invariant with respect to rigid motions of E^d.

Closely related to diameter and width of a convex body $K \in \mathcal{K}^d$ are circumradius $R(K)$ and inradius $r(K)$ ([BoF]). Clearly, $2r(K) \leq \Delta(K)$, $2R(K) \geq D(K)$ and on the other side by the theorems of JUNG [J] and STEINHAGEN [St]

$$R(K) \leq j_d \cdot D(K), \quad \text{with } j_d := \sqrt{\frac{d}{2d+2}}, \tag{1.1}$$

$$r(K) \geq s_d \cdot \Delta(K), \quad \text{with } s_d := \begin{cases} 1/(2\sqrt{d}), & \text{for odd } d \\ \sqrt{d+2}/(2d+2), & \text{for even } d. \end{cases} \tag{1.2}$$

Now, we can define successive circumradii and inradii in the same way we have defined successive diameters or widths and get the following four series of successive radii

Definition 1.2. For $K \in \mathcal{K}^d$ and $1 \leq i \leq d$ let

$$\textit{i)}\ R_i^\pi(K) := \min_{L \in \mathcal{L}_i^d} R(K|L), \qquad \textit{ii)}\ R_i^\sigma(K) := \min_{L \in \mathcal{L}_i^d} \max_{x \in L^\perp} R(K \cap (x+L)),$$

$$\textit{iii)}\ r_\pi^i(K) := \max_{L \in \mathcal{L}_i^d} r(K|L;L), \qquad \textit{iv)}\ r_\sigma^i(K) := \max_{L \in \mathcal{L}_i^d} \max_{x \in L^\perp} r(K \cap (x+L); x+L).$$

We obviously have $R_d^\pi(K) = R_d^\sigma(K) = R(K)$, $R_1^\pi(K) = R_1^\sigma(K) = \Delta(K)/2$, $r_\pi^d(K) = r_\sigma^d(K) = r(K)$ and $r_\pi^1(K) = r_\sigma^1(K) = D(K)/2$. If we replace in Definition 1.2 the first max-condition by a min-condition and vice versa we get four other series of successive radii, which now start with half of the diameter (half of the width) and terminate with the circumradius (inradius)

Definition 1.3. For $K \in \mathcal{K}^d$ and $1 \leq i \leq d$ let

$$\textit{i)}\ R_\pi^i(K) := \max_{L \in \mathcal{L}_i^d} R(K|L), \qquad \textit{ii)}\ R_\sigma^i(K) := \max_{L \in \mathcal{L}_i^d} \max_{x \in L^\perp} R(K \cap (x+L)),$$

$$\textit{iii)}\ r_i^\pi(K) := \min_{L \in \mathcal{L}_i^d} r(K|L;L), \qquad \textit{iv)}\ r_i^\sigma(K) := \min_{L \in \mathcal{L}_i^d} \max_{x \in L^\perp} r(K \cap (x+L); x+L).$$

It is easy to see that all these successive radii have the same properties as the successive diameters or widths.

Surprisingly, up to now there seems to be little literature on these series of successive diameters, widths and radii. Perhaps the first result is due to ZINDLER [Z] who showed that there is a 3-dimensional convex body K for which $R_2^\sigma(K) < R_2^\pi(K)$. PERELMANN [P] studied the quotient $R_i^\pi(K)/r_\sigma^{d-i+1}(K)$. Further they play a certain rôle in computational geometry (cf. e.g. [GK]). The only reference of successive diameters and successive widths known to us is by DAVIS [D], who posed the problem to determine the range of $D_i^\pi(K)/\Delta_\pi^{d-i+1}(K)$.

On the other hand apparently there is a rich theory for these functionals. For a first systematic study see HENK [He1], who showed e.g. that JUNG's inequality (1.1) can be generalized to the $R_\pi^i(K)$ and $R_\sigma^i(K)$ [He2]. In the same way there is a generalization of STEINHAGEN's inequality (1.2) to the $r_i^\pi(K)$, $r_i^\sigma(K)$, though the best constants are not known in all cases (compare [BHe1]). Further the successive diameters and widths are closely related to the well known successive minima from the geometry of numbers as it will be shown in [BHe2].

Here we mainly study inequalities between the volume and the successive diameters or widths (Theorem 2.1.). These are completely analogous to the classical second theorem of MINKOWSKI ([Mi], pp. 192, pp. 211, [GrL], pp. 59). The first result of this kind is by KUBOTA [Ku] who essentially proved the left side of Theorem 2.1. in dimension 2.

Next we transfer these results to the successive radii of Definition 1.2. and give the corresponding theorem for the radii of Definition 1.3. The existence of such inequalities was pointed out by J. M. WILLS (private communication). In fact this was the starting point of our investigation.

The third section generalizes the results for the well known intrinsic volumes though optimality is lost in some cases. In the final section we briefly study a different subject: A most challenging problem in the theory of numbers, geometry of numbers and convexity is — from different point of views — the study of $G(K)/V(K)$ for "large" bodies, where $G(\cdot)$ denotes the lattice point enumerator (see e.g. [GrL], [BW]).

From the convex point of view there is a satisfactory lower bound by the result of BOKOWSKI-HADWIGER-WILLS [BokHW], while there are only weak upper bounds. Application of the result in [BokHW] immediately gives an asymptotically sharp inequality of the form $G(K) > (1 - \frac{d}{2}\frac{1}{r(K)})V(K)$, $K \in \mathcal{K}^d$. The results of the third section together with known results give (apparently very weak) inequalities of the form $G(K) \leq \prod_{i=1}^{d}(1 + \frac{c_{id}}{D_i^\pi(K)})V(K)$. Nevertheless this gives a new approach to the problem and more direct studies might yield better results.

2. Volume, successive diameters, widths and radii

First we summarize some simple relations between the successive diameters, widths and successive radii, which will be used in the following.

Lemma 2.1. *Let $K \in \mathcal{K}^d$ and $L_j \in \mathcal{L}_j^d$. Then for $1 \leq i \leq j \leq d$*

$$\begin{aligned} &i) && D_i^\pi(K) \leq D_i^\pi(K|L_j; L_j) \leq D_{i+d-j}^\pi(K),\\ &ii) && \Delta_\pi^{i+d-j}(K) \leq \Delta_\pi^i(K|L_j; L_j) \leq \Delta_\pi^i(K)\\ &iii) && 2r_\pi^{d-i+1}(K) \leq \Delta_\pi^{d-i+1}(K) \leq D_i^\pi(K) \leq 2R_i^\pi(K). \end{aligned}$$

Up to the lower bounds in i) and ii) the same relations hold if the projections are replaced by sections.

Proof. The lower bound in i) and the upper bound in ii) are trivial. Let $L_{i+d-j} \in \mathcal{L}_{i+d-j}^d$. Then there exists an $L_i \in \mathcal{L}_i^d$ with $L_i \subset L_{i+d-j} \cap L_j$. Thus

$$D(K|L_{i+d-j}) \geq D((K|L_{i+d-j})|L_i) = D((K|L_j)|L_i) \geq D_i^\pi(K|L_j; L_j)$$

and

$$\begin{aligned} \Delta(K|L_{i+d-j}; L_{i+d-j}) &\leq \Delta((K|L_{i+d-j})|L_i; L_i) = \Delta((K|L_j)|L_i; L_i)\\ &\leq \Delta_i^\pi(K|L_j; L_j). \end{aligned}$$

This shows the upper bound in i) and the lower bound in ii).

Now, let $L_i \in \mathcal{L}_i^d$, $L_{d-i+1} \in \mathcal{L}_{d-i+1}^d$ with $\Delta_\pi^{d-i+1}(K) = \Delta(K|L_{d-i+1}; L_{d-i+1})$ and $D_i^\pi(K) = D(K|L_i)$. For an 1-dimensional subspace $L_1 \subset L_i \cap L_{d-i+1}$ we have

$$D(K|L_i) \geq D((K|L_i)|L_1) = \Delta((K|L_{d-i+1})|L_1; L_1) \geq \Delta(K|L_{d-i+1}; L_{d-i+1})$$

and this proves iii).

The proof of the inequalities where the projections are replaced by sections can be done in the same way. □

Now we can prove our main result

Theorem 2.1. *Let $K \in \mathcal{K}^d$. Then*

$$\frac{1}{d!} \cdot D_1^\pi(K) \cdot \ldots \cdot D_d^\pi(K) \leq V(K) \leq \Delta_\pi^1(K) \cdot \ldots \cdot \Delta_\pi^d(K), \tag{2.1}$$

$$\frac{1}{d!} \cdot D_1^\sigma(K) \cdot \ldots \cdot D_d^\sigma(K) \leq V(K) \leq \Delta_\sigma^1(K) \cdot \ldots \cdot \Delta_\sigma^d(K). \tag{2.2}$$

In general these bounds cannot be improved.

Proof. On account of $D_i^\sigma(K) \leq D_i^\pi(K)$, $\Delta_\sigma^i(K) \leq \Delta_\pi^i(K)$, $1 \leq i \leq d$, it suffices to prove the upper bound in (2.2) and the lower bound in (2.1). This will be done by induction with respect to the dimension. Obviously, for $d = 1$ all inequalities hold. So, let $d \geq 2$.

We start with the upper bound. By definition of the width there exist a $(d-1)$-dimensional linear subspace L_{d-1} and an unit vector $x \in L_{d-1}^\perp$, $\mu \in \mathbb{R}$, with

$$V(K) = \int_{-\Delta(K)/2}^{\Delta(K)/2} V(K \cap (tx + \mu x + L_{d-1}); tx + \mu x + L_{d-1}) dt.$$

After a suitable translation of K we may assume

$$V(K) \leq \Delta(K) \cdot V(K \cap L_{d-1}; L_{d-1}). \tag{2.3}$$

By the induction hypothesis we have

$$V(K) \leq \Delta(K) \cdot \Delta_\sigma^1(K \cap L_{d-1}; L_{d-1}) \cdot \ldots \cdot \Delta_\sigma^{d-1}(K \cap L_{d-1}; L_{d-1}),$$

and from Lemma 2.1. follows the upper bound in (2.2).

Next we show the lower bound. For that purpose let $x, y \in K$ be two points with $\|x - y\| = D(K)$, where $\|\cdot\|$ denotes the euclidean norm. By a suitable translation of K we may assume $y = -x$. Let L_{d-1} be the hyperplane with normal vector x. Steiner-symmetrization ([BoF], pp. 69) of K with respect to L_{d-1} yields a convex body $\overline{K}$ with $x, -x \in \overline{K}$, $V(\overline{K}) = V(K)$ and $\overline{K} \cap L_{d-1} = K|L_{d-1}$. Hence

$$V(\overline{K}) \geq V(\mathrm{conv}\{(\overline{K} \cap L_{d-1}), x, -x\}) = \frac{D(K)}{d} \cdot V(\overline{K} \cap L_{d-1}; L_{d-1}),$$

where $\mathrm{conv}\{\cdot\}$ denotes the convex hull. Thereby we have

$$V(K) \geq \frac{D(K)}{d} \cdot V(K|L_{d-1}; L_{d-1}). \tag{2.4}$$

On account of Lemma 2.1. the lower bound in (2.1) follows from the induction hypothesis.

To show that the bounds in general are best possible it suffices to consider the upper bound of (2.1) and the lower bound of (2.2). For $\mu > 0$ let $Q(\mu)$ be the rectangular parallelepipedon with edge lenghts $\mu, \mu^2, \dots, \mu^d$. Obviously, $D_i^\pi(Q(\mu))$ is not greater than the diameter of a rectangular parallelepipedon with edge lenghts $\mu, \mu^2, \dots, \mu^i$. Hence by Lemma 2.1. $\Delta_\pi^{d-i+1}(Q(\mu)) \le D_i^\pi(Q(\mu)) \le (\sum_{j=1}^i \mu^{2j})^{1/2}$. It follows

$$\frac{V(Q(\mu))}{\Delta_\pi^1(Q(\mu)) \cdot \ldots \cdot \Delta_\pi^d(Q(\mu))} \ge \frac{\mu\mu^2 \cdot \ldots \cdot \mu^d}{\mu(\sum_{j=1}^2 \mu^{2j})^{1/2} \cdot \ldots \cdot (\sum_{j=1}^d \mu^{2j})^{1/2}}.$$

For $\mu \to \infty$ the right hand side tends to 1 and this means that the upper bound in (2.1) is in general best possible.

Now, let e^j denote the j-th canonical unit vector and for $\mu > 0$ let $C(\mu)$ be the cross polytope with vertices $\pm\mu^j e^j$, $1 \le j \le d$. By Lemma 2.1. we have $D_i^\sigma(C(\mu)) \ge \Delta_\sigma^{d-i+1}(C(\mu))$ and obviously $\Delta_\sigma^{d-i+1}(C(\mu))$ is not less than the width of a cross polytope with vertices $\pm\mu^j e^j$, $i \le j \le d$. Hence we have $D_i^\sigma(C(\mu)) \ge 2(\sum_{j=i}^d \mu^{-2j})^{-1/2}$. It follows

$$\frac{V(C(\mu))}{D_1^\sigma(C(\mu)) \cdot \ldots \cdot D_d^\sigma(C(\mu))} \le \frac{1}{d!} \cdot \mu\mu^2 \cdot \ldots \cdot \mu^d \sqrt{\sum_{j=1}^d \mu^{-2j}} \sqrt{\sum_{j=2}^d \mu^{-2j}} \cdot \ldots \cdot \mu^{-d}.$$

For $\mu \to \infty$ the right hand side tends to $1/d!$ and this means that the lower bound in (2.2) is in general best possible. □

For the successive radii we can deduce from the above theorem the relations

Corollary 2.1. *Let $K \in \mathcal{K}^d$. Then*

$$(\textstyle\prod_{i=1}^d j_i) \cdot R_1^\pi(K) \cdot \ldots \cdot R_d^\pi(K) \le V(K) \le 2^d \cdot R_1^\pi(K) \cdot \ldots \cdot R_d^\pi(K), \tag{2.5}$$

$$(\textstyle\prod_{i=1}^d j_i) \cdot R_1^\sigma(K) \cdot \ldots \cdot R_d^\sigma(K) \le V(K) \le 2^d \cdot R_1^\sigma(K) \cdot \ldots \cdot R_d^\sigma(K), \tag{2.6}$$

$$(2^d/d!) \cdot r_\pi^1(K) \cdot \ldots \cdot r_\pi^d(K) \le V(K) \le (\textstyle\prod_{i=1}^d s_i) \cdot r_\pi^1(K) \cdot \ldots \cdot r_\pi^d(K), \tag{2.7}$$

$$(2^d/d!) \cdot r_\sigma^1(K) \cdot \ldots \cdot r_\sigma^d(K) \le V(K) \le (\textstyle\prod_{i=1}^d s_i) \cdot r_\sigma^1(K) \cdot \ldots \cdot r_\sigma^d(K), \tag{2.8}$$

where j_i and s_i are the constants from (1.1), (1.2). *In general the upper bounds of* (2.5), (2.6) *and the lower bounds in* (2.7), (2.8) *cannot be improved.*

Proof. The upper bounds in (2.5), (2.6) and the lower bounds in (2.7), (2.8) follow from Theorem 2.1. and Lemma 2.1. The extremal bodies in Theorem 2.1. show also that these bounds are best possible. The proof of the lower bounds of (2.5), (2.6) and the upper bounds in (2.7), (2.8) can be done in the same way as in Theorem 2.1. We only have to use the theorems of Jung (1.1) and Steinhagen (1.2) to estimate the the diameter by the circumradius in (2.4) and the width by the inradius in (2.3). □

Remarks.

(1) For $d = 2$ the lower bounds in (2.5) and (2.6) are best possible as a regular triangle shows.

(2) If K is a 0-symmetric convex body then we have $R(K) = D(K)/2$ and $r(K) = \Delta(K)/2$. This means that we can replace in the above Corollary $\prod_{i=1}^d j_i$ by $2^d/d!$ and $\prod_{i=1}^d s_i$ by 2^d. In this case all inequalities are best possible.

For the successive radii of Definition 1.3. we have the following general result

Theorem 2.2. *Let $K \in \mathcal{K}^d$. Then*

$$\kappa_d \cdot r_1^\pi(K) \cdot \ldots \cdot r_d^\pi(K) \le V(K) \le \kappa_d \cdot R_1^\pi(K) \cdot \ldots \cdot R_d^\pi(K), \tag{2.9}$$

$$\kappa_d \cdot r_1^\sigma(K) \cdot \ldots \cdot r_d^\sigma(K) \le V(K) \le \kappa_d \cdot R_1^\sigma(K) \cdot \ldots \cdot R_d^\sigma(K), \tag{2.10}$$

and for $K \in \mathcal{K}^d$ with nonempty interior equality holds iff K is a ball.

Proof. Obviously, if K is a ball all inequalities are satiesfied with equality. The upper bounds are trivial as ([BoF], p. 76)

$$V(K) \le \kappa_d (D(K)/2)^d \tag{2.11}$$

and $R_i^\pi(K) \ge R_i^\sigma(K) \ge R_1^\sigma(K) = D(K)/2$, $1 \le i \le d$. Now, if equality holds in one of the upper bounds we must have, on account of the foregoing relations, $R_i^\pi(K) = D(K)/2$ or $R_i^\sigma(K) = D(K)/2$ for $1 \le i \le d$, and so $V(K) = \kappa_d(R(K))^d$.

Since $r_i^\pi(K) \ge r_i^\sigma(K)$, $1 \le i \le d$, it suffices to prove the lower bound in (2.9). This will be done by induction with respect to the dimension. For $d = 1$ there is nothing to prove, so let $d \ge 2$. For the surface area $F(K)$ of K we have ([BoF], p. 48)

$$\kappa_{d-1} F(K) = \int_{S^{d-1}} V(K|H_u; H_u) du,$$

where S^{d-1} denotes the boundary of the d-dimensional unit ball and H_u denotes the hyperplane with normal vector u. By definition we have $r_i^\pi(K|H_u; H_u) \ge r_i^\pi(K)$, $1 \le i \le d-1$, and so

$$\kappa_{d-1} F(K) \ge d\kappa_d \kappa_{d-1} r_1^\pi(K) \cdot \ldots \cdot r_{d-1}^\pi(K).$$

On account of the trival relation $dV(K) \ge r(K)F(K)$ ([BoF], p. 38) we get the lower bound in (2.9).

Now suppose equality holds in the lower bound in (2.10). Then equality holds also in the lower bound in (2.9). By the previous proof we must have $dV(K) = r(K)F(K)$, $r_i^\pi(K|H_u; H_u) = r_i^\pi(K)$ and $V(K|H_u; H_u) = \kappa_{d-1} r_1^\pi(K|H_u; H_u) \cdot \ldots \cdot r_{d-1}^\pi(K|H_u; H_u)$ for all $u \in S^{d-1}$. By a simple inductive argument we can deduce that every projection of K onto a hyperplane is a $(d-1)$-dimensional ball with radius $\Delta(K)/2$. Hence K is a body of constant width and from $dV(K) = r(K)F(K)$ it follows that K is ball. □

Remarks.

(1) In general there is no lower (upper) bound for the volume with respect to the product of the radii $R_i^\pi(K)$ or $R_i^\sigma(K)$ ($r_\pi^i(K)$ or $r_\sigma^i(K)$) since the volume of a convex body may be arbitrary small (large) in proportion to the diameter (width).

(2) The obvious counterpart of (2.11) would be $V(K) \ge \kappa_d(\Delta(K))^d$ which is trivially wrong. Thus the lower bounds in Theorem 2.2. are natural counterparts of (2.11).

3. Intrinsic volumes, successive diameters and successive widths

The results of the previous section can be used to obtain analogous inequalities for the intrinsic volumes, though it turns out, that most results will probably not be best possible. The intrinsic volumes, introduced by McMullen [M], are normalized quermassintegrals and may be defined by the so-called Cauchy formula ([H], pp. 228, [MS])

$$V_i(K) = c_{id} \int V(K|L_i; L_i) dL_i, \quad 0 \le i \le d, \tag{3.1}$$

where the integration is with respect to the rotation group SO_d and dL_i denotes the rotation density of the i-dimensional linear subspace L_i. Further the constant c_{id} has to be chosen, such that for i-dimensional convex bodies $V_i(K)$ is just the i-dimensional volume of K.

Another way to introduce the V_i is by the concept of mixed volumes (cf. again [MS]): The volume of the linear combination $\lambda K + \mu M$, $\lambda, \mu \ge 0$, of two convex sets $K, M \subset E^d$ can be expressed in terms of the mixed volumes $V(K, i, M, d-i)$, $0 \le i \le d$, as a polynomial in λ and μ:

$$V(\lambda K + \mu M) = \sum_{i=0}^{d} \binom{d}{i} \lambda^i \mu^{d-i} V(K, i, M, d-i).$$

If the second body is a ball B the mixed volumes are, up to a factor, the intrinsic volumes $V(K, i, B, d-i) = \binom{d}{i}^{-1} \kappa_{d-i} V_i(K)$.

Taking account of the definition of the intrinsic volumes we only consider successive diameters or widths, which are defined via projections. To do this the following lemma will be useful

Lemma 3.1. (Cavalieri's principle for intrinsic volumes). *Let $K \in \mathcal{K}^d$ and L_{d-1} be a hyperplane with unit normal vector x. Then for $1 \le i \le d$*

$$V_i(K) \ge \int_{-\infty}^{\infty} V_{i-1}(K \cap (tx + L_{d-1})) dt.$$

Proof. The proof of the lemma follows immediately from a formula of Schneider [S] which yields in our special case:

$$\int_{-\infty}^{\infty} V_{i-1}(K \cap (tx + L_{d-1})) dt = \frac{\binom{d}{i}}{\kappa_{d-i}} V(K, i, \overline{B}, d-i),$$

where $\overline{B}$ denotes the ball of dimension $(d-1)$ in the hyperplane L_{d-1}. From this the assertion follows from the monotony of the mixed volumes (cf. [MS]). □

By this lemma we get

Theorem 3.1. *Let $K \in \mathcal{K}^d$. Then for $1 \le i \le d$*

$$\frac{1}{i!} \cdot D_d^{\pi}(K) \cdot \ldots \cdot D_{d-i+1}^{\pi}(K) \le V_i(K) \le \frac{\kappa_{id}}{c_{id}} \cdot \Delta_{\pi}^1(K) \cdot \ldots \cdot \Delta_{\pi}^i(K),$$

with $\kappa_{id} = \int dL_i$. Here the constant on the left hand side is best possible.

Proof. We start with the proof of the upper bound. For any i-dimensional plane $L_i \in \mathcal{L}_i^d$ we have by 2.1. $V(K|L_i;L_i) \le \Delta_\pi^1(K|L_i;L_i)\cdot\ldots\cdot\Delta_\pi^i(K|L_i;L_i)$. By Lemma 2.1. and (3.1) the assertion follows.

We prove the lower bound by induction with respect to the dimension. For $d=1$ the assertion is clear. For $K \subset E^d$ we choose $x, y \in K$, such that $\|x-y\| = D(K)$. Now let $\overline{K}$ be the body obtained by STEINER-symmetrization of K with respect to the hyperplane L_{d-1} orthogonal to $u = x-y/\|x-y\|$. Then we have $V_i(K) \ge V_i(\overline{K})$ and $\overline{K} \cap L_{d-1} = K|L_{d-1}$ ([BoF], pp. 69). Hence by Lemma 3.1. and Lemma 2.1.

$$
\begin{aligned}
V_i(K) \ge V_i(\overline{K}) &\ge \int_{-\infty}^{\infty} V_{i-1}(\overline{K} \cap (tu + L_{d-1})) \\
&\ge \int_{-D(K)/2}^{D(K)/2} \frac{(D(K)/2-|t|)^{i-1}}{(D(K)/2)^{i-1}} V_{i-1}(\overline{K}\cap L_{d-1})dt = \frac{D(K)}{i}\cdot V_{i-1}(K|L_{d-1}) \\
&\ge \frac{1}{i!}\cdot D(K)\cdot D_{d-1}^\pi(K|L_{d-1};L_{d-1})\cdot\ldots\cdot D_{d-i+1}^\pi(K|L_{d-1};L_{d-1}) \\
&\ge \frac{1}{i!}\cdot D_d^\pi(K)\cdot\ldots\cdot D_{d-i+1}^\pi(K).
\end{aligned}
$$

For $1 \le i \le d$ the i-dimensional crosspolytopes considered in the proof of Theorem 2.1. show that the lower bound cannot be improved. □

From Theorem 3.1. we may immediately deduce the following corollary which is needed in the next section and shows again the analogy of iterated diameters and successive minima (compare [He3])

Corollary 3.1. *Let $K \in \mathcal{K}^d$. Then for $0 \le i \le d$*

$$
\frac{c_{id}}{\kappa_{id} d!}\cdot D_1^\pi(K)\cdot\ldots\cdot D_{d-i}^\pi(K)\cdot V_i(K) \le V(K) \le i!\cdot\Delta_\pi^{i+1}(K)\cdot\ldots\cdot\Delta_\pi^d(K)\cdot V_i(K).
$$

Proof. From Theorem 1.1. and Lemma 2.1. we have

$$
V(K) \ge \frac{1}{d!} D_1^\pi(K)\cdot\ldots\cdot D_{d-i}^\pi(K)\cdot\Delta_\pi^1(K)\cdot\ldots\cdot\Delta_\pi^i(K).
$$

Together with the upper bound in Theorem 3.1. this shows the lower bound. The proof of the upper bound can be done in the same way. □

4. Applications to the lattice point enumerator

In this part of the paper we show some inequalities, which relate volume, lattice number and successive widths. Therefore let $\mathbb{Z}^d$ denote the set of all points with integral coordinates in E^d and for a convex body $K \in \mathcal{K}^d$ the lattice point enumerator $\mathrm{card}(K \cap \mathbb{Z}^d)$ is denoted by $G(K)$.

BOKOWSKI, HADWIGER and WILLS [BokHW] proved the following asymptotically tight inequality

$$
G(K) > V(K) - V_{d-1}(K). \tag{4.1}
$$

By $V_{d-1}(K) = F(K)/2 \leq dV(K)/(2r(K))$ we get from (4.1)

$$G(K) \geq \left(1 - \frac{d}{2}\frac{1}{r(K)}\right) V(K), \tag{4.2}$$

where the factor $d/2$ is best possible as a series of open lattice cubes — all vertices are lattice points — shows. Here it seems to be an interesting question, what happens if $r(K)$ is replaced by $\Delta(K)$.

On the other hand we have $G(K) \leq V(K + C^d)$, where C^d denotes the d-dimensional cube with edge length 1. This volume can be estimated in terms of the intrinsic volumes ([BW]) and we get

$$G(K) \leq \sum_{i=0}^{d} \kappa_{d-i}(\frac{\sqrt{d}}{2})^{d-i} V_i(K).$$

By Theorem 3.1 we have an upper bound of $V_i(K)$ in terms of the volume and certain successive diameters and hence we get an inequality of the form

$$G(K) \leq \prod_{i=1}^{d} \left(1 + \frac{c_{id}}{D_i^{\pi}(K)}\right) V(K), \tag{4.3}$$

where c_{id} are constants only depending on i and d. To get good constants seems to be a nontrivial problem. For a similar result see Wills [W] who proved $G(K) \leq (1 + \frac{\sqrt{d}}{2r(K)})^d V(K)$.

Altogether it seems to be an interesting open problem to get upper and lower bounds of the quotient $G(K)/V(K)$ in terms of the successive diameters, widths and radii.

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